Section 6.3  Medians and Altitudes of Triangles

Essential Question  What conjectures can you make about the medians and altitudes of a triangle?

**EXPLORATION 1**  Finding Properties of the Medians of a Triangle

**Work with a partner.**  Use dynamic geometry software. Draw any \(\triangle ABC\).

a. Plot the midpoint of \(BC\) and label it \(D\). Draw \(AD\), which is a **median** of \(\triangle ABC\). Construct the medians to the other two sides of \(\triangle ABC\).

b. What do you notice about the medians? Drag the vertices to change \(\triangle ABC\). Use your observations to write a conjecture about the medians of a triangle.

c. In the figure above, point \(G\) divides each median into a shorter segment and a longer segment. Find the ratio of the length of each longer segment to the length of the whole median. Is this ratio always the same? Justify your answer.

**EXPLORATION 2**  Finding Properties of the Altitudes of a Triangle

**Work with a partner.**  Use dynamic geometry software. Draw any \(\triangle ABC\).

a. Construct the perpendicular segment from vertex \(A\) to \(BC\). Label the endpoint \(D\). \(AD\) is an **altitude** of \(\triangle ABC\).

b. Construct the altitudes to the other two sides of \(\triangle ABC\). What do you notice?

c. Write a conjecture about the altitudes of a triangle. Test your conjecture by dragging the vertices to change \(\triangle ABC\).

**Communicate Your Answer**

3. What conjectures can you make about the medians and altitudes of a triangle?

4. The length of median \(RU\) in \(\triangle RST\) is 3 inches. The point of concurrency of the three medians of \(\triangle RST\) divides \(RU\) into two segments. What are the lengths of these two segments?
What You Will Learn

- Use medians and find the centroids of triangles.
- Use altitudes and find the orthocenters of triangles.

Using the Median of a Triangle

A **median of a triangle** is a segment from a vertex to the midpoint of the opposite side. The three medians of a triangle are concurrent. The point of concurrency, called the **centroid**, is inside the triangle.

**Theorem 6.7 Centroid Theorem**

The centroid of a triangle is two-thirds of the distance from each vertex to the midpoint of the opposite side.

The medians of \(\triangle ABC\) meet at point \(P\), and \(AP = \frac{2}{3}AE\), \(BP = \frac{2}{3}BF\), and \(CP = \frac{2}{3}CD\).

**CONSTRUCTION** Finding the Centroid of a Triangle

Use a compass and straightedge to construct the medians of \(\triangle ABC\).

**SOLUTION**

1. **Find midpoints**
   - Draw \(\triangle ABC\). Find the midpoints of \(AB\), \(BC\), and \(AC\). Label the midpoints of the sides \(D\), \(E\), and \(F\), respectively.

2. **Draw medians**
   - Draw \(AE\), \(BF\), and \(CD\). These are the three medians of \(\triangle ABC\).

3. **Label a point**
   - Label the point where \(AE\), \(BF\), and \(CD\) intersect as \(P\). This is the centroid.

**EXAMPLE 1** Using the Centroid of a Triangle

In \(\triangle RST\), point \(Q\) is the centroid, and \(SQ = 8\). Find \(QW\) and \(SW\).

**SOLUTION**

\[
SQ = \frac{2}{3}SW \\
8 = \frac{2}{3}SW \\
12 = SW
\]

Multiply each side by the reciprocal, \(\frac{3}{2}\).

Then \(QW = SW - SQ = 12 - 8 = 4\).

So, \(QW = 4\) and \(SW = 12\).
Finding the Centroid of a Triangle

Find the coordinates of the centroid of \( \triangle RST \) with vertices \( R(2, 1) \), \( S(5, 8) \), and \( T(8, 3) \).

**SOLUTION**

Step 1 Graph \( \triangle RST \).

Step 2 Use the Midpoint Formula to find the midpoint \( V \) of \( RT \) and sketch median \( SV \).

\[
V \left( \frac{2 + 8}{2}, \frac{1 + 3}{2} \right) = (5, 2)
\]

Step 3 Find the centroid. It is two-thirds of the distance from each vertex to the midpoint of the opposite side.

The distance from vertex \( S(5, 8) \) to \( V(5, 2) \) is \( 8 - 2 = 6 \) units.

So, the centroid is \( \frac{2}{3}(6) = 4 \) units down from vertex \( S \) on \( SV \).

So, the coordinates of the centroid \( P \) are \( (5, 8 - 4) \), or \( (5, 4) \).

### Monitoring Progress

There are three paths through a triangular park. Each path goes from the midpoint of one edge to the opposite corner. The paths meet at point \( P \).

1. Find \( PS \) and \( PC \) when \( SC = 2100 \) feet.
2. Find \( TC \) and \( BC \) when \( BT = 1000 \) feet.
3. Find \( PA \) and \( TA \) when \( PT = 800 \) feet.

Find the coordinates of the centroid of the triangle with the given vertices.

4. \( F(2, 5) \), \( G(4, 9) \), \( H(6, 1) \)

5. \( X(-3, 3) \), \( Y(1, 5) \), \( Z(-1, -2) \)

### Using the Altitude of a Triangle

An **altitude of a triangle** is the perpendicular segment from a vertex to the opposite side or to the line that contains the opposite side.

### Core Concept

**Orthocenter**

The lines containing the altitudes of a triangle are concurrent. This point of concurrency is the **orthocenter** of the triangle.

The lines containing \( AF \), \( BD \), and \( CE \) meet at the orthocenter \( G \) of \( \triangle ABC \).
As shown below, the location of the orthocenter $P$ of a triangle depends on the type of triangle.

**Acute triangle** $P$ is inside triangle.

**Right triangle** $P$ is on triangle.

**Obtuse triangle** $P$ is outside triangle.

### EXAMPLE 3  Finding the Orthocenter of a Triangle

Find the coordinates of the orthocenter of $\triangle XYZ$ with vertices $X(-5, -1)$, $Y(-2, 4)$, and $Z(3, -1)$.

**SOLUTION**

**Step 1** Graph $\triangle XYZ$.

**Step 2** Find an equation of the line that contains the altitude from $Y$ to $XZ$. Because $XZ$ is horizontal, the altitude is vertical. The line that contains the altitude passes through $Y(-2, 4)$. So, the equation of the line is $x = -2$.

**Step 3** Find an equation of the line that contains the altitude from $X$ to $YZ$.

The slope of $YZ$ is $\frac{-1 - 4}{3 - (-2)} = -1$.

Because the product of the slopes of two perpendicular lines is $-1$, the slope of a line perpendicular to $YZ$ is $1$. The line passes through $X(-5, -1)$.

$$y = mx + b$$

$$-1 = 1(-5) + b$$

$$4 = b$$

So, the equation of the line is $y = x + 4$.

**Step 4** Find the point of intersection of the graphs of the equations $x = -2$ and $y = x + 4$.

Substitute $-2$ for $x$ in the equation $y = x + 4$. Then solve for $y$.

$$y = x + 4$$

$$y = -2 + 4$$

$$y = 2$$

So, the coordinates of the orthocenter are $(-2, 2)$.

**Monitoring Progress**

Tell whether the orthocenter of the triangle with the given vertices is inside, on, or outside the triangle. Then find the coordinates of the orthocenter.

6. $A(0, 3), B(0, -2), C(6, -3)$
7. $J(-3, -4), K(-3, 4), L(5, 4)$
In an isosceles triangle, the perpendicular bisector, angle bisector, median, and altitude from the vertex angle to the base are all the same segment. In an equilateral triangle, this is true for any vertex.

**Example 4** Proving a Property of Isosceles Triangles

Prove that the median from the vertex angle to the base of an isosceles triangle is an altitude.

**Solution**

Given \( \triangle ABC \) is isosceles, with base \( \overline{AC} \).

\( BD \) is the median to base \( \overline{AC} \).

Prove \( BD \) is an altitude of \( \triangle ABC \).

**Paragraph Proof** Legs \( \overline{AB} \) and \( \overline{BC} \) of isosceles \( \triangle ABC \) are congruent. \( \overline{CD} \cong \overline{AD} \) because \( BD \) is the median to \( \overline{AC} \). Also, \( BD \cong BD \) by the Reflexive Property of Congruence (Thm. 2.1). So, \( \triangle ABD \cong \triangle CBD \) by the SSS Congruence Theorem (Thm. 5.8). \( \angle ADB \cong \angle CDB \) because corresponding parts of congruent triangles are congruent. Also, \( \angle ADB \) and \( \angle CDB \) are a linear pair. \( BD \) and \( AC \) intersect to form a linear pair of congruent angles, so \( BD \perp AC \) and \( BD \) is an altitude of \( \triangle ABC \).

Monitoring Progress

8. **What if?** In Example 4, you want to show that median \( BD \) is also an angle bisector. How would your proof be different?
1. **VOCABULARY** Name the four types of points of concurrency. Which lines intersect to form each of the points?

2. **COMPLETE THE SENTENCE** The length of a segment from a vertex to the centroid is __________ the length of the median from that vertex.

**Exercises 6.3**

**Vocabulary and Core Concept Check**

In Exercises 3–6, point \( P \) is the centroid of \( \triangle LMN \). Find \( PN \) and \( QP \). (See Example 1.)

3. \( QN = 9 \)

4. \( QN = 21 \)

5. \( QN = 30 \)

6. \( QN = 42 \)

In Exercises 7–10, point \( D \) is the centroid of \( \triangle ABC \). Find \( CD \) and \( CE \).

7. \( DE = 5 \)

8. \( DE = 11 \)

9. \( DE = 9 \)

10. \( DE = 15 \)

In Exercises 11–14, point \( G \) is the centroid of \( \triangle ABC \). \( BG = 6, AF = 12, \) and \( AE = 15 \). Find the length of the segment.

11. \( \overline{FC} \)

12. \( \overline{BF} \)

13. \( \overline{AG} \)

14. \( \overline{GE} \)

In Exercises 15–18, find the coordinates of the centroid of the triangle with the given vertices. (See Example 2.)

15. \( A(2, 3), B(8, 1), C(5, 7) \)

16. \( F(1, 5), G(−2, 7), H(−6, 3) \)

17. \( S(5, 5), T(11, −3), U(−1, 1) \)

18. \( X(1, 4), Y(7, 2), Z(2, 3) \)

In Exercises 19–22, tell whether the orthocenter is inside, on, or outside the triangle. Then find the coordinates of the orthocenter. (See Example 3.)

19. \( L(0, 5), M(3, 1), N(8, 1) \)

20. \( X(−3, 2), Y(5, 2), Z(−3, 6) \)

21. \( A(−4, 0), B(1, 0), C(−1, 3) \)

22. \( T(−2, 1), U(2, 1), V(0, 4) \)

**CONSTRUCTION** In Exercises 23–26, draw the indicated triangle and find its centroid and orthocenter.

23. isosceles right triangle

24. obtuse scalene triangle

25. right scalene triangle

26. acute isosceles triangle
**ERROR ANALYSIS** In Exercises 27 and 28, describe and correct the error in finding $DE$. Point $D$ is the centroid of $\triangle ABC$.

27. $DE = \frac{2}{3} AE$
   $DE = \frac{2}{3} (18)$
   $DE = 12$

   $\times$

28. $DE = \frac{2}{3} AD$
   $AD = 24$
   $DE = \frac{2}{3} (24)$
   $DE = 16$

   $\times$

**PROOF** In Exercises 29 and 30, write a proof of the statement. (See Example 4.)

29. The angle bisector from the vertex angle to the base of an isosceles triangle is also a median.

30. The altitude from the vertex angle to the base of an isosceles triangle is also a perpendicular bisector.

**CRITICAL THINKING** In Exercises 31–36, complete the statement with always, sometimes, or never. Explain your reasoning.

31. The centroid is ________ on the triangle.

32. The orthocenter is ________ outside the triangle.

33. A median is ________ the same line segment as a perpendicular bisector.

34. An altitude is ________ the same line segment as an angle bisector.

35. The centroid and orthocenter are ________ the same point.

36. The centroid is ________ formed by the intersection of the three medians.

37. **WRITING** Compare an altitude of a triangle with a perpendicular bisector of a triangle.

38. **WRITING** Compare a median, an altitude, and an angle bisector of a triangle.

39. **MODELING WITH MATHEMATICS** Find the area of the triangular part of the paper airplane wing that is outlined in red. Which special segment of the triangle did you use?

40. **ANALYZING RELATIONSHIPS** Copy and complete the statement for $\triangle DEF$ with centroid $K$ and medians $DH, EJ,$ and $FG$.
   a. $EJ = ____ KJ$
   b. $DK = ____ KH$
   c. $FG = ____ KF$
   d. $KG = ____ FG$

**MATHEMATICAL CONNECTIONS** In Exercises 41–44, point $D$ is the centroid of $\triangle ABC$. Use the given information to find the value of $x$.

41. $BD = 4x + 5$ and $BF = 9x$

42. $GD = 2x - 8$ and $GC = 3x + 3$

43. $AD = 5x$ and $DE = 3x - 2$

44. $DF = 4x - 1$ and $BD = 6x + 4$

45. **MATHEMATICAL CONNECTIONS** Graph the lines on the same coordinate plane. Find the centroid of the triangle formed by their intersections.
   $y_1 = 3x - 4$
   $y_2 = \frac{3}{4}x + 5$
   $y_3 = -\frac{3}{2}x - 4$

46. **CRITICAL THINKING** In what type(s) of triangles can a vertex be one of the points of concurrency of the triangle? Explain your reasoning.
47. **WRITING EQUATIONS** Use the numbers and symbols to write three different equations for PE.

48. **HOW DO YOU SEE IT?** Use the figure.

49. **MAKING AN ARGUMENT** Your friend claims that it is possible for the circumcenter, incenter, centroid, and orthocenter to all be the same point. Do you agree? Explain your reasoning.

50. **DRAWING CONCLUSIONS** The center of gravity of a triangle, the point where a triangle can balance on the tip of a pencil, is one of the four points of concurrency. Draw and cut out a large scalene triangle on a piece of cardboard. Which of the four points of concurrency is the center of gravity? Explain.

51. **PROOF** Prove that a median of an equilateral triangle is also an angle bisector, perpendicular bisector, and altitude.

52. **THOUGHT PROVOKING** Construct an acute scalene triangle. Find the orthocenter, centroid, and circumcenter. What can you conclude about the three points of concurrency?

53. **CONSTRUCTION** Follow the steps to construct a nine-point circle. Why is it called a nine-point circle?

54. **PROOF** Prove the statements in parts (a)–(c).

55. **Determining whether \( AB \) is parallel to \( CD \).** (Section 3.5)

56. **A(5, 6), B(−1, 3), C(−4, 9), D(−16, 3)\)

57. **A(−3, 6), B(5, 4), C(−14, −10), D(−2, −7)\)

58. **A(6, −3), B(5, 2), C(−4, −4), D(−5, 2)\)

59. **A(−5, 6), B(−7, 2), C(7, 1), D(4, −5)\)